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Approximation of Mappings with Values Which Are Upper Semicontinuous Functions

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In this paper we study some aspects of the approximation of mappings taking values in a special class of upper semicontinuous functions. Some Korovkin type theorems for positive linear operators are obtained, and consequences of these theorems for a special class of operators defined through partial sum stochastic processes are analyzed. © 2001 Elsevier Science

Key Words: function valued map; fuzzy random variable; linear operator; partial sum stochastic process; positive operator; random set; upper semicontinuous function.

1. INTRODUCTION

The aim of this paper is to introduce some aspects of the approximation of mappings whose images are functions of a certain class. Let $\mathscr{F}(R^p)$ be the subspace of $[0, 1]^{R^p}$, which consists of those functions which are upper semicontinuous and have compact support. A first Korovkin type theorem is obtained for positive linear operators on the class of continuous mappings from a compact Hausdorff space into $\mathscr{F}(R^p)$, equipped with a uniform type metric.

It can be observed that the class of non-empty compact subsets of R^p can be embedded into the class of mappings we are considering by taking the indicator function of the set, and so results in the paper are valid for compact (not necessarily convex) random sets.

On the basis of this result we show how to obtain other Korovkin type theorems by considering topologies on $\mathscr{F}(R^p)$ weaker than that of the above-mentioned metric.

The main results of the paper are contained in Sections 3 and 4. In the former, Korovkin type theorems and applications of these are presented. In Section 4, the above results allow us to study the convergence of some

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operators based on partial sum stochastic processes. Concepts and results which are required in order to develop the paper are collected in Section 2.

2. PRELIMINARIES

Let $\mathscr{K}(R^p)$ denote the class of non-empty compact subsets of R^p, let $\mathscr{K}_c(R^p)$ be the subclass of non-empty compact convex subsets of R^p, and let B denote the ball $\{x \in R^p: |x| \leqslant 1\}$, where $|\cdot|$ is the Euclidean norm on R^p. The convex hull of a set A will be denoted co A. The Euclidean inner product will be denoted $\langle \cdot, \cdot \rangle$. If A is a subset of R^p we will denote by I_A the indicator function of A.

The space $\mathscr{K}(R^p)$ will be endowed with a linear structure given by the Minkowski addition and the product by a scalar; that is,

$$A+C=\{a+c \mid a \in A, c \in C\}, \qquad \lambda A=\{\lambda a \mid a \in A\},$$

for all $A, C \in \mathscr{K}(R^p)$, and $\lambda \in R$. The space $(\mathscr{K}(R^p), +, .)$ is not a vector space.

If $A \in \mathscr{K}_c(R^p)$ and $\lambda_1, \lambda_2 \in [0, +\infty)$ then $\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2) A$. Moreover, $\mathscr{K}_c(R^p)$ is closed under Minkowski addition and product by a scalar.

Given $A, C \in \mathscr{K}(R^p)$, the Hausdorff distance between A and C is defined by

$$d_H(A, C) = \max\{\sup_{a \in A} \inf_{c \in C} |a-c|, \sup_{c \in C} \inf_{a \in A} |a-c|\}.$$

Then $(\mathscr{K}(R^p), d_H)$ is a complete, separable metric space and $(\mathscr{K}_c(R^p), d_H)$ is a closed subspace (see Debreu [5]).

If $A \in \mathscr{K}(R^p)$, its *magnitude* is defined to be

$$\|A\|_{\mathscr{K}} = d_H(\{0\}, A) = \sup_{x \in A} |x|.$$

Some well-known properties of this metric are:

(i) $d_H(A+C, D+E) \leqslant d_H(A, D) + d_H(C, E)$,

(ii) $d_H(aA, bA) \leqslant |a-b| \, \|A\|_{\mathscr{K}}$,

(iii) $d_H(\text{co } A, \text{co } C) \leqslant d_H(A, C)$,

(iv) $d_H(A, C) = \inf\{\varepsilon > 0 \mid A \subset C + \varepsilon B, C \subset A + \varepsilon B)$,

where $A, C, D, E \in \mathscr{K}(R^p)$, $a, b \in R$.

Let $(\Omega, \mathscr{A})$ be a measurable space. A set valued function $X: \Omega \to \mathscr{K}(R^p)$ is called a *random set* if it is $\mathscr{A}$-$\mathscr{B}_{d_H}$ measurable, where $\mathscr{B}_{d_H}$ denotes the Borel σ-field in $\mathscr{K}(R^p)$.

If $X: \Omega \to \mathcal{K}(R^p)$ is a random set, its *magnitude* is defined to be the mapping $\|X\|_{\mathcal{K}}: \Omega \to R$ with $\|X\|_{\mathcal{K}}(\omega) = \|X(\omega)\|_{\mathcal{K}}$ for all $\omega \in \Omega$. The measurability of a random set X implies that $\|X\|_{\mathcal{K}}$ is measurable (see Hiai and Umegaki [8]).

If $X, T: \Omega \to \mathcal{K}(R^p)$ are random sets, we will denote by $D_H(X, Y)$ the value $\sup_{\omega \in \Omega} d_H(X(\omega), Y(\omega))$.

It is well known that if $X, Y: \Omega \to \mathcal{K}(R^p)$ are random sets, then $X+Y$, λX and co X, where $(\text{co } X)(\omega) = \text{co}(X(\omega))$ for all $\omega \in \Omega$, are also random sets for all $\lambda \in R$ (see Matheron [13]).

$\mathcal{F}(R^p)$ is the class of upper semicontinuous functions $V = R^p \to [0, 1]$ such that supp V is compact. Given $V \in \mathcal{F}(R^p)$ we define its α-level set $V_\alpha = \{x \in R^p \mid V(x) \geqslant \alpha\}$ if $\alpha > 0$ and $V_0 = \text{supp } V$. We will denote by $\mathcal{F}_c(R^p)$ the subclass of $\mathcal{F}(R^p)$ such that $V \in \mathcal{F}_c(R^p)$ if and only if V_α is convex for all $\alpha \in [0, 1]$.

The class $\mathcal{F}(R^p)$ can be endowed with a linear structure, for which addition and product by a scalar are defined by

$$(U+V)(x) = \sup\{\alpha \in [0, 1] \mid x \in U_\alpha + V_\alpha\}$$

$$(\lambda U)(x) = \begin{cases} U(\lambda^1 x) & \text{if } \lambda \neq 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases}$$

for all $U, V \in \mathcal{F}(R^p)$, $\lambda \in R$.

It is possible to see that these operations are inherited levelwise from those defined on $\mathcal{K}(R^p)$; that is, for all $\alpha \in [0, 1]$

$$(\lambda U)_\alpha = \lambda U_\alpha \qquad \text{and} \qquad (U+V)_\alpha = U_\alpha + V_\alpha$$

and $\mathcal{F}(R^p)$ and $\mathcal{F}_c(R^p)$ are closed under them (Puri and Ralescu [17]).

$\mathcal{F}(R^p)$ can be endowed with the d_∞ metric where

$$d_\infty(V, W) = \sup_{\alpha \in [0, 1]} d_H(V_\alpha, W_\alpha).$$

$(\mathcal{F}(R^p), d_\infty)$ is a complete metric space (see Puri and Ralescu [18]), but it is not separable (see Klement *et al.* [11]).

$\mathcal{F}_c(R^p)$ will be embedded into the Banach space $L^\infty(S^{p-1} \times [0, 1])$ of bounded, measurable real functions on $S^{p-1} \times [0, 1]$ by means of the mapping $s: \mathcal{F}_c(R^p) \to L^\infty(S^{p-1} \times [0, 1])$ with $V \mapsto s(\cdot, \cdot, V): S^{p-1} \times [0, 1] \to R$ and $s(r, \alpha, V) = \sup_{a \in V_\alpha} \langle r, a \rangle$, for all $V \in \mathcal{F}_c(R^p)$ and $(r, \alpha) \in S^{p-1} \times [0, 1]$.

From the properties of the Hausdorff metric it follows that

(i) $d_\infty(A+C, D+E) \leqslant d_\infty(A, D) + d_\infty(C, E)$,

(ii) $d_\infty(aA, bA) \leqslant |a-b| \, \|A_0\|_{\mathscr{K}}$,

(iii) $d_\infty(\text{co } A, \text{co } C) \leqslant d_\infty(A, C)$,

where $A, C, D, E \in \mathscr{F}(R^p)$, $a, b \in R$, and if $V \in \mathscr{F}(R^p)$, then $\text{co } V \in \mathscr{F}_c(R^p)$ is the upper semicontinuous mapping such that $(\text{co } V)_\alpha = \text{co}(V_\alpha)$ for all $\alpha \in [0, 1]$.

In order to state a property similar to the fourth one of d_H we consider a partial ordering on $\mathscr{F}(R^p)$. Thus, for $V, W \in \mathscr{F}(R^p)$ we will write $V \subset W$ if $V_\alpha \subset W_\alpha$ for all $\alpha \in [0, 1]$ or, equivalently, $V(x) \leqslant W(x)$ for all $x \in R^p$. Thus,

(iv) $d_\infty(A, C) = \inf\{\varepsilon > 0 \mid A \subset C + \varepsilon I_B, C \subset A + \varepsilon I_B\}$.

A mapping $X: \Omega \to \mathscr{F}(R^p)$ will be said to be *measurable*, if the α-level function $X_\alpha: \Omega \to \mathscr{K}(R^p)$, with $X_\alpha(\omega) = (X(\omega))_\alpha$ for all $\omega \in \Omega$, is a random set for all $\alpha \in [0, 1]$. This kind of mapping is also referred to as a *fuzzy random variable* (see Puri and Ralescu [18]).

Given a probability space $(\Omega, \mathscr{A}, P)$, a measurable mapping $X: \Omega \to \mathscr{F}(R^p)$ is said to be *integrably bounded with respect to* P, if and only if $\|X_0\|_{\mathscr{K}} \in L^1(P)$. In this case, the *expected value* of X with respect to P, denoted by $\int X \, dP$ is the unique element V of $\mathscr{F}(R^p)$ such that V_α is the Aumann integral of X_α with respect to P for all $\alpha \in [0, 1]$ (see Aumann [3] and Puri and Ralescu [18]).

From now on Ω will denote a compact Hausdorff topological space and we will consider its Borel σ-field. If (E, d) is a metric space, $\mathscr{C}(\Omega, E)$ will denote the set of all continuous mappings $X: \Omega \to E$, and $\mathscr{B}(\Omega, E)$ will denote the set of all bounded mappings $X: \Omega \to E$.

If $X, Y: \Omega \to \mathscr{F}(R^p)$ are measurable mappings, then $X+Y$, λX and $\text{co } X$ are also measurable for all $\lambda \in R$. These properties are inherited from the corresponding properties of random sets.

It is clear to see that if $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$, then for each $\alpha \in [0, 1]$ the mapping $X_\alpha: \Omega \to \mathscr{K}(R^p)$ is continuous, and hence X is measurable.

On the class $\mathscr{C}(\Omega, \mathscr{F}(R^p))$ we consider the D_∞ metric defined by

$$D_\infty(X, Y) = \sup_{t \in \Omega} d_\infty(X(t), Y(t)).$$

The order on $\mathscr{F}(R^p)$ induces one on $\mathscr{C}(\Omega, \mathscr{F}(R^p))$: if $X, Y \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$, then $X \subset Y$ will mean that $X(t) \subset Y(t)$ for all $t \in \Omega$. Then

$$D_\infty(X, Y) = \inf\{\varepsilon > 0 \mid X \subset Y + \varepsilon I_B, Y \subset X + \varepsilon I_B\}.$$

Given $X \in \mathscr{B}(\Omega, \mathscr{F}(R^p))$ we will denote $\sup_{x \in \Omega} \|X_0\|_{\mathscr{K}}(x)$ by $\|X\|_{\mathscr{C}}$.

A mapping $T\colon \mathscr{C}(\Omega, \mathscr{F}(R^p)) \to \mathscr{C}(\Omega, \mathscr{F}(R^p))$ will be said to be *linear*, if $T(aX+bY) = aTX+bTY$ for all $a, b \in [0, \infty)$ and $X, Y \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$, it will be called *sublinear*, if $T(X+Y) \subset TX+TY$ and $T(aX) = aTX$ for all $a \in [0, \infty)$ and $X, Y \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$, it will be called *positive*, if $TX \subset TY$ for all $X, Y \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$ with $X \subset Y$ and it will be called an $\mathscr{F}$*-operator*, if $(TX)(\omega) = (TY)(\omega)$ for all $X, Y \in C(\Omega, \mathscr{F}(R^p))$ and $\omega \in \Omega$ such that $X(\omega) = Y(\omega)$.

If $T\colon \mathscr{C}(\Omega, \mathscr{F}(R^p)) \to \mathscr{C}(\Omega, \mathscr{F}(R^p))$ is a positive sublinear operator, then

$$D_\infty(TX, TY) \leqslant \|TI_B\|_{\mathscr{C}}\, D_\infty(X, Y).$$

3. KOROVKIN TYPE THEOREMS ON $\mathscr{C}(\Omega, \mathscr{F}(R^p))$

This section contains the main results of the paper. It is devoted to some Korovkin type theorems for positive linear operators on the class $\mathscr{C}(\Omega, \mathscr{F}(R^p))$. These results will be essential to the development of Section 4. Notice that if $A \in \mathscr{F}(R^p)$, then A will also denote the constant function in $\mathscr{C}(\Omega, \mathscr{F}(R^p))$ which has value A.

THEOREM 3.1. *Let Ω be a compact Hausdorff topological space, and let $\phi\colon \Omega^2 \to R$ be a continuous mapping such that $\phi(x, x) = 0$ for all $x \in \Omega$ and $\phi(x, y) > 0$ for all $x, y \in \Omega$ with $x \neq y$. Let $L_{n,\lambda}\colon \mathscr{C}(\Omega, \mathscr{F}(R^p)) \to \mathscr{B}(\Omega, \mathscr{F}(R^p))$, $n \in N$, $\lambda \in \Lambda$ (Λ being an index set), be positive linear operators, such that there exists $n_0 \in N$ with*

$$\sup_{\lambda \in \Lambda,\, n \geqslant n_0} \|L_{n,\lambda} I_B\|_{\mathscr{C}} < \infty.$$

Let $T\colon \mathscr{C}(\Omega, \mathscr{F}(R^p)) \to \mathscr{C}(\Omega, \mathscr{F}(R^p))$ be an $\mathscr{F}$-operator.

Then the following conditions are equivalent:

(i) *$D_\infty(L_{n,\lambda}X, TX) \to 0$ uniformly in $\lambda \in \Lambda$ for all $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$,*

(ii) *$D_\infty(L_{n,\lambda}(\phi(x,\cdot)\, I_B), T(\phi(x,\cdot)\, I_B)) \to 0$ uniformly in $\lambda \in \Lambda$ for all $x \in \Omega$, $D_\infty(L_{n,\lambda}A, TA) \to 0$ uniformly in $\lambda \in \Lambda$ for all $A \in \mathscr{F}(R^p)$,*

(iii) *$\sup_{x \in \Omega} \|(L_{n,\lambda}(\phi(x,\cdot)\, I_B))_0\|_{\mathscr{K}}\, (x) \to 0$ uniformly in $\lambda \in \Lambda$, $D_\infty(L_{n,\lambda}A, TA) \to 0$ uniformly in $\lambda \in \Lambda$ for all $A \in \mathscr{F}(R^p)$.*

Proof. Obviously (i) implies (ii) because of the continuity of ϕ.

To show that (ii) implies (iii) note that given $\varepsilon > 0$, since ϕ is continuous, for all $(x, t) \in \Omega^2$ there exists a neighborhood $V_{x,t}$ of (x, t) such that $|\phi(y, z) - \phi(w, u)| \leqslant \varepsilon$ for all $(y, z), (w, u) \in V_{x,t}$. Since we consider the product topology in Ω^2, we can take $V_{x,t} = V_t(x) \times W_x(t)$ with $V_t(x)$ a neighborhood of x depending on t and $W_x(t)$ a neighborhood of t depending on x.

Since Ω is compact, the cover $\{W_x(t)\}_{t \in \Omega}$ of Ω has a finite subcover $\{W_x(t_i)\}_{i=1}^{p(x)}$. Then $V(x) = \bigcap_{i=1}^{p(x)} V_{t_i}(x)$ is a neighborhood of x for each $x \in \Omega$, and thus $\{V(x)\}_{x \in \Omega}$ has a finite subcover $\{V(x_j)\}_{j=1}^{s}$.

Given $w, y \in V(x)$, for each $t \in \Omega$ there exists $i_0 \leqslant p(x)$ such that $t \in W_x(t_{i_0})$, and so $(w, t), (y, t) \in V_{t_{i_0}}(x) \times W_x(t_{i_0})$ which implies that $|\phi(w, t) - \phi(y, t)| \leqslant \varepsilon$ for all $t \in \Omega$. Then, because of the linearity and positivity of $L_{n,\lambda}$,

$$\begin{aligned} D_\infty(L_{n,\lambda}(\phi(w,\cdot)\, I_B), L_{n,\lambda}(\phi(y,\cdot)\, I_B)) &\leqslant \|L_{n,\lambda} I_B\|_{\mathscr{C}}\, D_\infty(\phi(w,\cdot)\, I_B, \phi(y,\cdot)\, I_B) \\ &= \|\phi(w,\cdot) - \phi(y,\cdot)\|\, \|L_{n,\lambda} I_B\|_{\mathscr{C}} \leqslant \varepsilon\, \|L_{n,\lambda} I_B\|_{\mathscr{C}} \end{aligned}$$

for all $\lambda \in \Lambda$ and $n \in N$.

Given $x \in \Omega$, since $\{V(x_j)\}_{j=1}^{s}$ is a cover of the set Ω, there exists $j_0 \leqslant s$ with $x \in V(x_{j_0}) = \bigcap_{i=1}^{p(x_{j_0})} V_{t_i}(x_{j_0})$, on the other hand, since $\{W_{x_{j_0}}(t_i)\}_{i=1}^{p(x_{j_0})}$ is a cover of Ω, there exists $i_0 \leqslant p(x_{j_0})$ with $x \in W_{x_{j_0}}(t_{i_0})$, then $(x, x), (x_{j_0}, x) \in V_{t_{i_0}}(x_{j_0}) \times W_{x_{j_0}}(t_{i_0})$ and so

$$d_\infty(L_{n,\lambda}(\phi(x,\cdot)\, I_B)(x), L_{n,\lambda}(\phi(x_{j_0},\cdot)\, I_B)(x)) \leqslant \varepsilon\, \|L_{n,\lambda} I_B\|_{\mathscr{C}}.$$

In accordance with this,

$$\begin{aligned} & d_\infty(L_{n,\lambda}(\phi(x,\cdot)\, I_B)(x), T(\phi(x,\cdot)\, I_B)(x)) \\ &\quad \leqslant d_\infty(L_{n,\lambda}(\phi(x,\cdot)\, I_B)(x), L_{n,\lambda}(\phi(x_{j_0},\cdot)\, I_B)(x)) \\ &\qquad + d_\infty(L_{n,\lambda}(\phi(x_{j_0},\cdot)\, I_B)(x), T(\phi(x_{j_0},\cdot)\, I_B)(x)) \\ &\qquad + d_\infty(T(\phi(x_{j_0},\cdot)\, I_B)(x), T(\phi(x,\cdot)\, I_B)(x)) \\ &\quad \leqslant \varepsilon\, \|L_{n,\lambda} I_B\|_{\mathscr{C}} + d_\infty(L_{n,\lambda}(\phi(x_{j_0},\cdot)\, I_B)(x)\, T(\phi(x_{j_0},\cdot)\, I_B)(x)) + \varepsilon\, \|T I_B\|_{\mathscr{C}}. \end{aligned}$$

Since T is an $\mathscr{F}$-operator, $(TX)(x) = T(X(x))(x)$ for all $x \in \Omega$ and $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$, and so

$$T(\phi(x,\cdot)\, I_B)(x) = T(\phi(x, x)\, I_B)(x) = T(I_{\{0\}})(x) = \lim_{n \to \infty} (L_{n,\lambda} I_{\{0\}})(x) = I_{\{0\}}$$

for any $\lambda \in \Lambda$.

Then

$$\begin{aligned} d_\infty(L_{n,\lambda}(\phi(x,\cdot)\, I_B)(x), T(\phi(x,\cdot)\, I_B)(x)) &= d_\infty(L_{n,\lambda}(\phi(x,\cdot)\, I_B)(x), I_{\{0\}}) \\ &= \|(L_{n,\lambda}(\phi(x,\cdot)\, I_B))_0\|_{\mathscr{K}}\ (x) \end{aligned}$$

and so

$$\sup_{\lambda \in \Lambda} \sup_{x \in \Omega} \|(L_{n,\lambda}(\phi(x,\cdot)\, I_B))_0\|_{\mathscr{K}}\,(x)$$
$$\leqslant \max_{1 \leqslant j \leqslant s} \sup_{\lambda \in \Lambda} D_\infty(L_{n,\lambda}(\phi(x_j,\cdot)\, I_B), T(\phi(x_j,\cdot)\, I_B))$$
$$+\varepsilon(\sup_{\lambda \in \Lambda} \|L_{n,\lambda} I_B\|_{\mathscr{C}} + \|T I_B\|_{\mathscr{C}}),$$

which proves that (ii) implies (iii) since $\sup_{\lambda \in \Lambda, n \geqslant n_0} \|L_{n,\lambda} I_B\|_{\mathscr{C}} < \infty$ for some n_0.

With respect to (iii) implies (i), given $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$ with $X \neq I_{\{0\}}$ and $\varepsilon > 0$ (the case $X = I_{\{0\}}$ is trivial), for each $x \in \Omega$ there exists an open neighborhood V_x of x such that $d_\infty(X(x), X(y)) \leqslant \varepsilon$ for all $y \in V_x$.

Let $O = \bigcup_{x \in \Omega} (V_x \times V_x)$. The complement O^c of O in Ω^2 is compact.

Given $(x, y) \in \Omega^2$, if $(x, y) \in O$ then there exists $z \in \Omega$ such that $(x, y) \in V_z \times V_z$, which implies that $d_\infty(X(x), X(y)) \leqslant 2\varepsilon$.

If $(x, y) \notin O$, then $\phi(x, y) > 0$ and, by its continuity, ϕ reaches a minimum $M > 0$ in O^c. Then $d_\infty(X(x), X(y)) \leqslant 2\,\|X\|_{\mathscr{C}}\, M^{-1}\phi(x, y)$.

Then $d_\infty(X(x), X(y)) \leqslant 2\varepsilon + 2\,\|X\|_{\mathscr{C}}\, M^{-1}\phi(x, y)$ for all $(x, y) \in \Omega^2$, so, given $x \in \Omega$, the constant function $X(x)$ satisfies the relation

$$X(x) \subset X + 2\varepsilon I_B + 2\,\|X\|_{\mathscr{C}}\, M^{-1}\phi(x,\cdot)\, I_B,$$

and, because of the linearity and positivity of $L_{n,\lambda}$,

$$L_{n,\lambda}(X(x))(x) \subset (L_{n,\lambda}X)(x) + 2\varepsilon(L_{n,\lambda}I_B)(x) + 2\,\|X\|_{\mathscr{C}}\, M^{-1} L_{n,\lambda}(\phi(x,\cdot)\, I_B)(x).$$

In the same way

$$(L_{n,\lambda}X)(x) \subset L_{n,\lambda}(X(x))(x) + 2\varepsilon(L_{n,\lambda}I_B)(x) + 2\,\|X\|_{\mathscr{C}}\, M^{-1} L_{n,\lambda}(\phi(x,\cdot)\, I_B)(x),$$

and so

$$d_\infty(L_{n,\lambda}(X(x))(x), (L_{n,\lambda}X)(x))$$
$$\leqslant 2\varepsilon\, \|(L_{n,\lambda}I_B)_0\|_{\mathscr{K}}\,(x) + 2\,\|X\|_{\mathscr{C}}\, M^{-1} \sup_{\lambda \in \Lambda} \|(L_{n,\lambda}(\phi(x,\cdot)\, I_B)_0\|_{\mathscr{K}}\,(x).$$

We know that

$$\sup_{\lambda \in \Lambda} D_\infty(L_{n,\lambda}X, TX) \leqslant \sup_{\lambda \in \Lambda} \sup_{x \in \Omega} d_\infty((L_{n,\lambda}X)(x), L_{n,\lambda}(X(x))(x))$$
$$+ \sup_{\lambda \in \Lambda} \sup_{x \in \Omega} d_\infty(L_{n,\lambda}(X(x))(x), (TX)(x)).$$

By the previous inequality, we only have to show that the second term also tends to 0.

Since T is an $\mathscr{F}$-operator,

$$\sup_{\lambda \in \Lambda} \sup_{x \in \Omega} d_\infty(L_{n,\lambda}(X(x))(x), (TX)(x))$$

$$= \sup_{\lambda \in \Lambda} \sup_{x \in \Omega} d_\infty(L_{n,\lambda}(X(x))(x), T(X(x))(x))$$

$$\leqslant \sup_{\lambda \in \Lambda} \sup_{x \in \Omega} D_\infty(L_{n,\lambda}(X(x)), T(X(x))).$$

If

$$\sup_{\lambda \in \Lambda} \sup_{x \in \Omega} D_\infty(L_{n,\lambda}(X(x)), T(X(x)))$$

does not tend to zero, then there exist $\varepsilon > 0$ and $\{x_n\}$ such that

$$D_\infty(L_{n,\lambda}(X(x_n)), T(X(x_n))) > \varepsilon.$$

The functions X and TX are continuous, $X(\Omega)$ and $TX(\Omega)$ are compact, and the sequence $\{x_n\}$ has a limit point x. So it is possible to choose a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $D_\infty(X(x_{n_k}), X(x)) < \frac{1}{k}$ and $D_\infty((TX)(x_{n_k}), (TX)(x)) < \frac{1}{k}$.

Then

$$\varepsilon < \sup_{\lambda \in \Lambda} D_\infty(L_{n_k \lambda}(X(x_{n_k})), (TX)(x_{n_k}))$$

$$\leqslant \sup_{\lambda \in \Lambda} D_\infty(L_{n_k \lambda}(X(x_{n_k})), L_{n_k \lambda}(X(x)))$$

$$+ \sup_{\lambda \in \Lambda} D_\infty(L_{n_k \lambda}(X(x)), T(X(x))) + D_\infty(T(X(x)), (TX)(x_{n_k})).$$

On the other hand

$$\sup_{\lambda \in \Lambda} D_\infty(L_{n_k \lambda}(X(x_{n_k})), L_{n_k \lambda}(X(x))) \leqslant \sup_{\lambda \in \Lambda} \|L_{n_k \lambda} I_B\|_{\mathscr{C}} D_\infty(X(x_{n_k}), X(x)) \to 0.$$

By hypothesis

$$\sup_{\lambda \in \Lambda} D_\infty(L_{n_k \lambda}(X(x)), T(X(x))) \to 0,$$

and since T is an $\mathscr{F}$-operator,

$$D_\infty(T(X(x)), T(X(x_{n_k}))) = \sup_{y \in \Omega} d_\infty(T(X(x))(y), T(X(x_{n_k}))(y))$$

$$= d_\infty((TX)(x), (TX)(x_{n_k})) \to 0,$$

which concludes the proof of the theorem. ∎

Remark 3.1. To prove that (iii) implies (i) it is enough to assume that $\phi: \Omega^2 \to R$ is bounded, non-negative and such that $\phi(x, \cdot) \in \mathscr{C}(\Omega, R)$ for all $x \in \Omega$ and $\inf \phi(F) > 0$ for all closed subsets F of Ω^2 with $F \cap \{(x, x), x \in \Omega\} = \varnothing$. Thus, the case (iii) implies (i) generalizes a theorem in Nishishiraho [14] (see [1, Theorem 4.2.10]).

Remark 3.2. Let $\hat{T}: \mathscr{F}(R^p) \to \mathscr{F}(R^p)$ be a continuous mapping, and define the operator $T: \mathscr{C}(\Omega, \mathscr{F}(R^p)) \to \mathscr{C}(\Omega, \mathscr{F}(R^p))$ by $TX = \hat{T} \circ X$. Then T is an $\mathscr{F}$-operator. As an example consider $\hat{T}: \mathscr{F}(R^p) \to \mathscr{F}(R^p)$ with $\hat{T}V = \operatorname{co} V$.

Remark 3.3. In Theorem 3.1 the hypothesis of convergence for constants in (ii) and (iii) cannot be dropped. The two conditions

(i) $D_\infty(L_{n,\lambda}X, TX) \to 0$ uniformly in $\lambda \in \Lambda$ for all $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$,

(ii′) $D_\infty(L_{n,\lambda}(\phi(x, \cdot)\, I_B), T(\phi(x, \cdot)\, I_B)) \to 0$ uniformly in $\lambda \in \Lambda$ for all $x \in \Omega$,

are not equivalent. It is sufficient to take $L_{n,\lambda}$ to be the identity for all $n \in N$ and $\lambda \in \Lambda$ and $TX = \operatorname{co} X$. The equivalence of the previous conditions would imply that $X = \operatorname{co} X$ for all $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$.

Remark 3.4. A result similar to that in Theorem 3.1 can be obtained for operators $L_{n,\lambda}: \mathscr{G} \to \mathscr{B}(\Omega, \mathscr{F}(R^p))$, $\mathscr{G}$ being a subclass of $\mathscr{C}(\Omega, \mathscr{F}(R^p))$ closed under addition and multiplication by an scalar and satisfying the conditions

— for all $x \in \Omega$, $\phi(x, \cdot)\, I_B \in \mathscr{G}$,

— if $X \in \mathscr{G}$, then for each $x \in \Omega$ the constant function $X(x)$ belongs to $\mathscr{G}$.

In this case the condition of convergence for constants is required only for those in $\mathscr{G}$.

Remark 3.5. It is noteworthy that $\mathscr{F}(R^p)$ is not a convex cone in the sense given by Prolla [15, 16] and Keimel and Roth [9]. Although $\mathscr{F}_c(R^p)$ is a metric convex cone in the sense of [15, 16], the application of Theorem 3.1 to $\mathscr{F}_c(R^p)$-valued continuous mappings is not a consequence of Prolla's results, since we do not require the operators to be monotonically regular.

As an example of Theorem 3.1 we mention the following one adapted from Altomare and Campiti [1, pp. 224–225]:

(1) Let $\Omega = [a, b]$, $a, b \in R$, $a < b$, $\phi(x, y) = (x - y)^2$ and Λ a singleton. Then it is easy to deduce that the following conditions are equivalent:

(i) $D_\infty(L_n X, TX) \to 0$ for all $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$,

(ii) $D_\infty(L_n(e_i I_B), T(e_i I_B)) \to 0$, for $i = 0, 1, 2$, where $e_i(t) = t^i$, $D_\infty(L_n A, TA) \to 0$ for all $A \in \mathscr{F}(R^p)$.

Take T to be the identity. Considering $\mathscr{G} = \{f I_B \mid f \in \mathscr{C}(\Omega, R)\}$ in Remark 3.4, we obtain the classical Korovkin theorem [12]. Considering $\mathscr{G} = \{I_F \mid F \in \mathscr{C}(\Omega, \mathscr{K}(R^p))\}$ we generalize the Korovkin type theorem in Vitale [20].

(2) If (Ω, d) is a compact metric space, $u: [0, \infty) \to [0, \infty)$ is strictly increasing continuous with $u(0) = 0$, and $\phi(x, y) = u(d(x, y))$, then ϕ satisfies the conditions of Theorem 3.1.

Remark 3.6. With only minor modifications in the proof, Theorem 3.1 can be slightly improved in the following ways:

— linearity can be replaced by sublinearity,

— instead of the limit $\mathscr{F}$-operator T (the same for all $\lambda \in \Lambda$), a family $\{T_\lambda\}_{\lambda \in \Lambda}$ of limit $\mathscr{F}$-operators can be taken without losing uniform convergence. The family $\{T_\lambda\}_{\lambda \in \Lambda}$ must be such that $\{T_\lambda X\}_\lambda$ is equicontinuous for all $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$ and $\sup_{\lambda \in \Lambda} \|T_\lambda I_B\|_{\mathscr{C}} < \infty$.

These technical extensions will be needed for the proof of Theorem 3.2 and some of the subsequent examples.

The following result will be useful in order to obtain Korovkin type theorems when we consider some topologies in $\mathscr{F}(R^p)$ weaker than the one induced by d_∞.

THEOREM 3.2. *Let Ω be a compact Hausdorff topological space, and let $\phi: \Omega^2 \to R$ be a continuous mapping such that $\phi(x, x) = 0$ for all $x \in \Omega$ and $\phi(x, y) \neq 0$ for all $x, y \in \Omega$ with $x \neq y$. Let $L_{n,\lambda}: \mathscr{C}(\Omega, \mathscr{F}(R^p)) \to \mathscr{C}(\Omega, \mathscr{F}(R^p))$, $n \in N$, $\lambda \in \Lambda$ (Λ being an index set), be positive linear operators, such that there exists $n_0 \in N$ with*

$$\sup_{\lambda \in \Lambda, n \geqslant n_0} \|L_{n,\lambda} I_B\|_{\mathscr{C}} < \infty.$$

Let $T: \mathscr{C}(\Omega, \mathscr{F}(R^p)) \to \mathscr{C}(\Omega, \mathscr{F}(R^p))$ be an $\mathscr{F}$-operator. Let $\hat{S}_i: \mathscr{F}(R^p) \to \mathscr{F}(R^p)$ $i \in I$ (I being an index set) be a family of sublinear positive mappings such that

$$\sup_{i \in I} \|S_i I_B\|_{\mathscr{C}} < \infty,$$

and define $S_i\colon \mathscr{C}(\Omega, \mathscr{F}(R^p)) \to \mathscr{C}(\Omega, \mathscr{F}(R^p))$ *by* $S_i X = \hat{S}_i \circ X$.
Then the following conditions are equivalent:

(i) $D_\infty(S_i L_{n,\lambda} X, S_i T X) \to 0$ *uniformly in* $(\lambda, i) \in \Lambda \times I$ *for all* $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$,

(ii) $D_\infty(S_i L_{n,\lambda}(\phi(x,\cdot)\, I_B), (S_i T)(\phi(x,\cdot)\, I_B)) \to 0$ *uniformly in* $(\lambda, i) \in \Lambda \times I$ *for all* $x \in \Omega$, $D_\infty(S_i L_{n,\lambda} A, S_i T A) \to 0$ *uniformly in* $(\lambda, i) \in \Lambda \times I$ *for all* $A \in \mathscr{F}(R^p)$,

(iii) $\sup_{x \in \Omega} \|(S_i L_{n,\lambda}(\phi(x,\cdot)\, I_B))_0\|_{\mathscr{K}}\, (x) \to 0$ *uniformly in* $(\lambda, i) \in \Lambda \times I$ $D_\infty(S_i L_{n,\lambda} A, S_i T A) \to 0$ *uniformly in* $(\lambda, i) \in \Lambda \times I$ *for all* $A \in \mathscr{F}(R^p)$.

If in each of the conditions (i), (ii), (iii) *"uniformly in* $(\lambda, i) \in \Lambda \times I$*" is replaced by "uniformly in* $\lambda \in \Lambda$ *for each* $i \in I$*" then the three conditions which result are equivalent.*

Proof. We should remark that the operators $S_i L_{n,\lambda}$ and $S_i T$ satisfy the conditions

— $S_i L_{n,\lambda}$ are positive and sublinear,
— $\sup_{i \in I} \|S_i T I_B\|_{\mathscr{C}} < \infty$,
— $\{S_i T X\}_{i \in I}$ is equicontinuous for all $X \in \mathscr{C}(\Omega, \mathscr{F}(R^p))$ since

$$d_\infty((S_i T X)(x), (S_i T X)(y)) \leqslant \|\hat{S}_i I_B\|_{\mathscr{K}}\, d_\infty((TX)(x), (TX)(y)),$$

— $S_i T$ is an $\mathscr{F}$-operator for all $i \in I$,
— there exists $n_0 \in N$ such that for all $n \geqslant n_0$

$$\sup_{n \geqslant n_0,\, i \in I,\, \lambda \in \Lambda} \|S_i L_{n,\lambda} I_B\|_{\mathscr{C}} < \infty$$

since

$$\sup_{n \geqslant n_0,\, i \in I,\, \lambda \in \Lambda} \|S_i L_{n,\lambda} I_B\|_{\mathscr{C}} \leqslant \sup_{n \geqslant n_0,\, i \in I,\, \lambda \in \Lambda} \|S_i I_B\|_{\mathscr{C}}\, \|L_{n,\lambda} I_B\|_{\mathscr{C}}.$$

With respect to the first part we only have to take Remark 3.6 into account, considering $\Lambda' = \Lambda \times I$, the set of operators $\{S_i L_{n,\lambda}\}_{i \in I,\, n \in N,\, \lambda \in \Lambda}$, and as limit $\mathscr{F}$-operators the family $\{S_i T\}_{i \in I}$.

To obtain the second part, it is sufficient to apply the first part to each singleton class $\{S_i T\}$ with $i \in I$. ∎

As examples of Theorem 3.2 we can consider the following situations:

(1) Let us consider the family $\{\hat{S}_\alpha\}_{0\leqslant\alpha\leqslant 1}$ with $\hat{S}_\alpha\colon \mathscr{F}(R^p)\to\mathscr{F}(R^p)$ given by $\hat{S}_\alpha V = I_{V_\alpha}$. Obviously this is a family of linear positive operators and $\sup_{\alpha\in[0,1]} \|S_\alpha I_B\|_{\mathscr{C}} = 1$.

In this way we obtain a Korovkin type theorem for the topology generated by the α-level mappings and so the following conditions are equivalent:

(i) $D_H((L_{n,\lambda}X)_\alpha, (TX)_\alpha)\to 0$ uniformly in $\lambda\in\Lambda$ for all $\alpha\in[0,1]$ and for all $X\in\mathscr{C}(\Omega,\mathscr{F}(R^p))$,

(ii) $D_H((L_{n,\lambda}(\phi(x,\cdot)\, I_B))_\alpha, (T(\phi(x,\cdot)\, I_B))_\alpha)\to 0$ uniformly in $\lambda\in\Lambda$ for all $\alpha\in[0,1]$ and for all $x\in\Omega$, $D_H((L_{n,\lambda}A)_\alpha, (TA)_\alpha)\to 0$ uniformly in $\lambda\in\Lambda$ for all $\alpha\in[0,1]$ and for all $A\in\mathscr{F}(R^p)$,

(iii) $\sup_{x\in\Omega} \|(L_{n,\lambda}(\phi(x,\cdot)\, I_B))_0\|_{\mathscr{K}}\,(x)\to 0$ uniformly in $\lambda\in\Lambda$, $D_H((L_{n,\lambda}A)_\alpha, (TA)_\alpha)\to 0$ uniformly in $\lambda\in\Lambda$ for all $\alpha\in[0,1]$ and for all $A\in\mathscr{F}(R^p)$,

(2) Let $\{\rho_i\}_{i\in I}$, $\rho_i\colon\mathscr{F}(R^p)\to R$ be a family of positive semimagnitudes (i.e. $\rho(V)\geqslant 0$, $\rho(\lambda V)=\lambda\rho(V)$, $\rho(U+V)\leqslant\rho(U)+\rho(V)$, and if $U\subset V$ then $\rho(U)\leqslant\rho(V)$ for $U, V\in\mathscr{F}(R^p)$ and $\lambda\in[0,\infty)$) such that $\sup_{i\in I}\rho_i(I_B)<\infty$. Define $\{\hat{S}_i\}_{i\in I}$ with $\hat{S}_i\colon\mathscr{F}(R^p)\to\mathscr{F}(R^p)$ by $\hat{S}_i V=\rho_i(V)\, I_B$. Then $\{\hat{S}_i\}_{i\in I}$ satisfies the hypothesis of Theorem 3.2 and by its second part we obtain a Korovkin type theorem for the topology generated by the family $\{\rho_i\}_{i\in I}$, and so the following conditions are equivalent:

(i) $\|\rho_i(L_{n,\lambda}X)-\rho_i(TX)\|\to 0$ uniformly in $\lambda\in\Lambda$ for all $i\in I$ and for all $X\in\mathscr{C}(\Omega,\mathscr{F}(R^p))$,

(ii) $\|\rho_i(L_{n,\lambda}(\phi(x,\cdot)\, I_B))-\rho_i(T(\phi(x,\cdot)\, I_B))\|\to 0$ uniformly in $\lambda\in\Lambda$ for all $i\in I$ and for all $x\in\Omega$, $\|\rho_i(L_{n,\lambda}A)-\rho_i(TA)\|\to 0$ uniformly in $\lambda\in\Lambda$ for all $i\in I$ and for all $A\in\mathscr{F}(R^p)$,

(iii) $\sup_{x\in\Omega}\|\rho_i(L_{n,\lambda}(\phi(x,\cdot)))(x)\|\to 0$ uniformly in $\lambda\in\Lambda$ for all $i\in I$, $\|\rho_i(L_{n,\lambda}A)-\rho_i(TA)\|\to 0$ uniformly in $\lambda\in\Lambda$ for all $i\in I$ and for all $A\in\mathscr{F}(R^p)$.

(3) As a particular case of (2), we can consider the family $\{\rho_{r,\alpha}\}_{(r,\alpha)\in S^{p-1}\times[0,1]}$ with $\rho_{r,\alpha}\colon\mathscr{F}_{cl}(R^p)\to R$ given by $\rho_{r,\alpha}V=s(r,\alpha,V)$ ($\mathscr{F}_{cl}(R^p)$ being the subclass of $\mathscr{F}_c(R^p)$ such that $V\in\mathscr{F}_{cl}(R^p)$, if and only if the mapping $P_V\colon \alpha\in[0,1]\mapsto V_\alpha$ is Lipschitz), so we obtain convergence in the w-s topology (see Xue *et al.* [21])

(4) Analogously, we can consider the family $\{\sup_{\alpha\in[0,1]} s(r,\alpha,\cdot)\}_{r\in S^{p-1}}$ defined on $\mathscr{F}_c(R^p)$, and then we obtain the pointwise convergence of the support function uniformly in α. It is possible to obtain the same result for $\mathscr{K}_c(R^p)$ by embedding this class in $\mathscr{F}_c(R^p)$.

4. AN APPLICATION TO THE CONVERGENCE OF OPERATORS BASED ON PARTIAL SUMS

In this section we apply the preceding results in the study of uniform convergence of a particular class of operators defined through a stochastic process.

Let $\mathscr{B}_0$ be the class of those Borel subsets of $[0, \infty)^d$ having non-zero Lebesgue measure, and denote by ∂A the boundary of $A \in \mathscr{B}_0$. If $\mathscr{A} \subset \mathscr{B}_0$, $\delta > 0$, then we set

$$r_{\mathscr{A}}(\delta) = \sup_{A \in \mathscr{A}} m_d(\partial A + \delta B),$$

where m_d is Lebesgue measure in R^d. We will say that $\mathscr{A}$ satisfies the *smooth boundary condition* if $r_{\mathscr{A}}(\delta) \to 0$ as $\delta \to 0^+$ (notice that the term "smooth" is not used in the sense that ∂A is a smooth curve for $A \in \mathscr{A}$).

Let Φ be the family of sequences $\varphi = \{\varphi_n\}_n \subset \mathscr{B}_0$ such that $m_d(\varphi_n) \to \infty$. For the sake of brevity, we will say that $\Psi \subset \Phi$ satisfies the *smooth boundary condition* if $\{n^{-1}\varphi_n \mid \varphi \in \Psi, n \in N\}$ does.

The following "strong law of large volumes" is a restatement of Theorem 1 and Remark 1 in [4]:

THEOREM 4.1 (Bass–Pyke uniform strong law of large numbers). *Let ξ be a random variable such that $E|\xi| < \infty$, and let $\{\xi_j\}_{j \in N^d}$ be a d-dimensional array of independent random variables having the same distribution as ξ. Let $\Psi \subset \Phi$ satisfy the smooth boundary condition and be such that $\sup_{\varphi \in \Psi} m_d(\varphi_n)^{-1} = O(n^{-d})$. Then,*

$$\sup_{\varphi \in \Psi} \left| m_d(\varphi_n)^{-1} \sum_{j \in \varphi_n \cap N^d} \xi_j - E[\xi] \right| \to 0$$

almost surely.

Let $I \subset R$ be an Interval. Let $\{\xi_{j,x}\}_{j \in N^d, x \in I}$ be a doubly indexed stochastic process such that for each $x \in I$ all $\xi_{j,x}$ are independent and identically distributed. Moreover, for each $x \in I$ assume that $E[\xi_{\vec{1}x}] = x$ (where $\vec{1} = (1, 1, ..., 1)$) and denote by $\sigma^2(x)$ the variance $\operatorname{Var}[\xi_{\vec{1}x}]$.

Given a sequence $\varphi \in \Phi$, we define a process $\{S^{\varphi}_{n,x}\}_{n \in N, x \in I}$ by

$$S^{\varphi}_{n,x} = m_d(\varphi_n)^{-1} \sum_{j \in \varphi_n \cap N^d} \xi_{j,x}.$$

From now on, we will assume that the values of $S^{\varphi}_{n,x}$ lie in I.

We denote by $\mathscr{B}(I, \mathscr{F}(R^p))$ and $\mathscr{C}_b(I, \mathscr{F}(R^p))$, respectively, the classes of bounded and bounded continuous $\mathscr{F}(R^p)$ valued functions on I. Now we are finally able to define sequence based operators $L_n^\varphi : \mathscr{C}_b(I, \mathscr{F}(R^p)) \to \mathscr{B}(I, \mathscr{F}(R^p))$ by

$$L_n^\varphi(X, x) = \int X \, dP_{n,x}^\varphi,$$

$P_{n,x}^\varphi$ being the probability distribution on I induced by $S_{n,x}^\varphi$. Obviously, these operators are linear and positive, since the Aumann integral has both properties under the conditions considered.

For each $x \in I$, let Q_n^x be the nth convolution with itself of the distribution induced by $\xi_{\bar{1}x}$. In the following theorem, we use Theorem 3.1 and its Example 1 in order to provide sufficient conditions for the uniform convergence of a family of operators L_n^φ.

THEOREM 4.2. *Let $I \subset R$ be an interval and let $\Omega \subset I$ be compact. Let $\Psi \subset \Phi$ satisfy the smooth boundary condition and be such that $\sup_{\varphi \in \Psi} m_d(\varphi_n)^{-1} = O(n^{-d})$. Moreover, assume that $\sup_{x \in \Omega} \sigma^2(x) < \infty$ and $\sup\{Q_n^x(Z) \mid Z$ is an atom of $Q_n^x\}$ tends to 0 uniformly in $x \in \Omega$. Then, for each bounded continuous mapping $X : I \to \mathscr{F}(R^p)$,*

$$L_n^\varphi(X, \cdot) \xrightarrow{D_\infty} \text{co}\, X$$

uniformly in $\varphi \in \Psi$, where $D_\infty(Y, Z) = \sup_{x \in \Omega} d_\infty(Y(x), Z(x))$.

Proof. One only needs to apply the above-mentioned example by taking $\Lambda = \Psi$. In order to reach the conclusion, one checks convergence for the family $\{I_B, e_1 I_B, e_2 I_B\} \cup \mathscr{F}(R^p)$.

(1) Let us see whether $\sup_{\varphi \in \Psi} D_\infty(L_n^\varphi(I_B, \cdot), I_B)$ tends to 0. In fact,

$$L_n^\varphi(I_B, x) = \int I_B \, dP_{n,x}^\varphi = \int I_B \, dP = I_B,$$

so it is actually trivial. Notice that the Change-of-Variable Theorem for the Aumann integral can be applied since functions under the integral sign take values in $\mathscr{F}_c(R^p)$.

(2) We shall prove that $\sup_{\varphi \in \Psi} D_\infty(L_n^\varphi(e_1 I_B, \cdot), e_1 I_B)$ tends to 0. As before,

$$L_n^\varphi(e_1 I_B, x) = \int e_1 I_B \, dP_{n,x}^\varphi = \int S_{n,x}^\varphi I_B \, dP = E[S_{n,x}^\varphi] \, I_B.$$

But

$$E[S_{n,x}^\varphi] = \int m_d(\varphi_n)^{-1} \sum_{j \in \varphi_n \cap N^d} \xi_{j,x} \, dP = m_d(\varphi_n)^{-1} \operatorname{card}(\varphi_n \cap N^d)\, x,$$

and so

$$\begin{aligned}\sup_{\varphi\in\Psi} D_\infty(L_n^\varphi(e_1 I_B,\cdot), e_1 I_B) &= \sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} d_\infty(m_d(\varphi_n)^{-1}\operatorname{card}(\varphi_n\cap N^d)\, xI_B, xI_B)\\ &= \sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} |m_d(\varphi_n)^{-1}\operatorname{card}(\varphi_n\cap N^d)\, x - x|\\ &\leqslant (\sup\Omega) \sup_{\varphi\in\Psi} |m_d(\varphi_n)^{-1}\operatorname{card}(\varphi_n\cap N^d) - 1|.\end{aligned}$$

Finally, observe that an application of Theorem 4.1 to an array of degenerate random variables, each of constant value 1, yields that

$$\sup_{\varphi\in\Psi} |m_d(\varphi_n)^{-1}\operatorname{card}(\varphi_n\cap N^d) - 1| \to 0$$

and therefore

$$\sup_{\varphi\in\Psi} D_\infty(L_n^\varphi(e_1 I_B,\cdot), e_1 I_B) \to 0.$$

(3) We shall prove that $\sup_{\varphi\in\Psi} D_\infty(L_n^\varphi(e_2 I_B,\cdot), e_2 I_B)$ tends to 0. We see that

$$\begin{aligned}L_n^\varphi(e_2 I_B, x) &= \int e_2 I_B\, dP_{n,x}^\varphi = \int (S_{n,x}^\varphi)^2 I_B\, dP = E[(S_{n,x}^\varphi)^2]\, I_B\\ &= (\operatorname{Var}[S_{n,x}^\varphi] + E[S_{n,x}^\varphi]^2)\, I_B,\end{aligned}$$

and so

$$\begin{aligned}\sup_{\varphi\in\Psi} D_\infty(L_n^\varphi(e_2 I_B,\cdot), e_2 I_B) &= \sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} d_\infty((\operatorname{Var}[S_{n,x}^\varphi] + E[S_{n,x}^\varphi]^2)\, I_B, x^2 I_B)\\ &= \sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} |\operatorname{Var}[S_{n,x}^\varphi] + E[S_{n,x}^\varphi]^2 - x^2|\\ &\leqslant \sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} \operatorname{Var}[S_{n,x}^\varphi] + \sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} |E[S_{n,x}^\varphi]^2 - x^2|.\end{aligned}$$

Now,

$$\begin{aligned}&\sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} \operatorname{Var}[S_{n,x}^\varphi]\\ &\quad= \sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} \operatorname{Var}\left[m_d(\varphi_n)^{-1} \sum_{j\in\varphi_n\cap N^d} \xi_{j,x}\right]\\ &\quad= \sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} m_d(\varphi_n)^{-2}\operatorname{card}(\varphi_n\cap N^d)\,\sigma^2(x)\\ &\quad\leqslant \Big(1 + \sup_{\varphi\in\Psi} |m_d(\varphi_n)^{-1}\operatorname{card}(\varphi_n\cap N^d) - 1|\Big) \sup_{x\in\Omega}\sigma^2(x) \sup_{\varphi\in\Psi} m_d(\varphi_n)^{-1}.\end{aligned}$$

This vanishes at infinity, since the first factor tends to 1 (as seen in the second step of the proof), the second one is bounded by hypothesis, and the third one is $O(n^{-d})$ by hypothesis.

Finally,

$$\sup_{\varphi \in \Psi} \sup_{x \in \Omega} |E[S^{\varphi}_{n,x}]^2 - x^2|$$

$$= \sup_{\varphi \in \Psi} \sup_{x \in \Omega} |m_d(\varphi_n)^{-2} \operatorname{card}(\varphi_n \cap N^d)^2 x^2 - x^2|$$

$$\leqslant \max\{\inf \Omega, \sup \Omega\}^2 \sup_{\varphi \in \Psi} |m_d(\varphi_n)^{-2} \operatorname{card}(\varphi_n \cap N^d)^2 - 1|$$

and

$$\sup_{\varphi \in \Psi} |m_d(\varphi_n)^{-2} \operatorname{card}(\varphi_n \cap N^d)^2 - 1|$$

$$= \sup_{\varphi \in \Psi} |m_d(\varphi_n)^{-1} \operatorname{card}(\varphi_n \cap N^d) - 1|$$

$$\times (\sup_{\varphi \in \Psi} |m_d(\varphi_d(\varphi_n)^{-1} \operatorname{card}(\varphi_n \cap N^d) + 1|)$$

$$\leqslant \sup_{\varphi \in \Psi} |m_d(\varphi_n)^{-1} \operatorname{card}(\varphi_n \cap N^d) - 1|$$

$$\times (\sup_{\varphi \in \Psi} |m_d(\varphi_n)^{-1} \operatorname{card}(\varphi_n \cap N^d) - 1| + 2),$$

which vanishes at infinity.

(4) Let us see whether $\sup_{\varphi \in \Psi} D_\infty(L^{\varphi}_n(A, \cdot), \operatorname{co} A)$ tends to 0, for all $A \in \mathscr{F}(R^p)$. Notice that the change-of-variable technique in the first step of the proof is no longer valid.

We decompose each probability measure $P^{\varphi}_{n,x}$ into its atomic part $(P^{\varphi}_{n,x})_a$ and its atomless part $(P^{\varphi}_{n,x})_b$. We define atoms $P^{\varphi}_{n,x} = \{Z \mid Z \text{ is an atom of } P^{\varphi}_{n,x}\}$. Then

$$P^{\varphi}_{n,x} = \lambda^{\varphi}_{n,x}(P^{\varphi}_{n,x})_a + (1 - \lambda^{\varphi}_{n,x})(P^{\varphi}_{n,x})_b,$$

where $\lambda^{\varphi}_{n,x} = P^{\varphi}_{n,x}(\bigcup \text{atoms } P^{\varphi}_{n,x})$. Then

$$\int A \, dP^{\varphi}_{n,x} = \sum_{Z \in \text{atoms } P^{\varphi}_{n,x}} P^{\varphi}_{n,x}(Z) \, A + (1 - \lambda^{\varphi}_{n,x}) \operatorname{co} A.$$

In case some $P^{\varphi}_{n,x}$ has infinitely many atoms, the series above is defined to be the d_∞-limit of the sequence of partial sums (this is an obvious generalization of the arguments in [7, Section 7]). It now follows that

$$\begin{aligned}
d_\infty&\left(\int A\,dP^{\varphi}_{n,x}, \operatorname{co} A\right)\\
&= d_\infty\Bigg(\sum_{Z\in \text{atoms } P^{\varphi}_{n,x}} P^{\varphi}_{n,x}(Z)\,A+(1-\lambda_{n,x})\operatorname{co} A,\\
&\qquad \lambda_{n,x}\operatorname{co} A+(1-\lambda_{n,x})\operatorname{co} A\Bigg)\\
&\leqslant d_\infty\left(\sum_{Z\in \text{atoms } P^{\varphi}_{n,x}} P^{\varphi}_{n,x}(Z)\,A, \lambda_{n,x}\operatorname{co} A\right)\\
&= d_\infty\left(\sum_{Z\in \text{atoms } P^{\varphi}_{n,x}} P^{\varphi}_{n,x}(Z)\,A, \sum_{Z\in \text{atoms } P^{\varphi}_{n,x}} \operatorname{co}(P^{\varphi}_{n,x}(Z)\,A)\right).
\end{aligned}$$

Now an application of the Shapley–Folkman inequality (see for instance [2]) and a limiting argument (if $P^{\varphi}_{n,x}$ has infinitely many atoms) yield straightforwardly the inequality

$$d_\infty\left(\int A\,dP^{\varphi}_{n,x}, \operatorname{co} A\right) \leqslant \|A_0\| \sup\{P^{\varphi}_{n,x}(Z) \mid Z \text{ is an atom of } P^{\varphi}_{n,x}\}.$$

For the sake of brevity, we set $a(P)=\sup\{P(Z) \mid Z$ is an atom of $P\}$, whenever P is a probability measure. Accordingly,

$$\sup_{\varphi\in\Psi} D_\infty(L^{\varphi}_n(A,\cdot), \operatorname{co} A) \leqslant \|A_0\| \sup_{\varphi\in\Psi}\ \sup_{x\in\Omega} a(P^{\varphi}_{n,x}).$$

Recall that Q^x_m denotes the mth convolution with itself of the distribution of $\xi_{\vec{1}x}$, and note that $P^{\varphi}_{n,x}$ and $Q^x_{\operatorname{card}(\varphi_n\cap N^d)}$ are induced by random variables differing only by a scale factor. Indeed, $P^{\varphi}_{n,x}$ is the distribution induced by $S^{\varphi}_{n,x}=m_d(\varphi_n)^{-1}\sum_{j\in\varphi_n\cap N^d}\xi_{j,x}$ whereas $Q^x_{\operatorname{card}(\varphi_n\cap N^d)}$ is induced by $\sum_{j\in\varphi_n\cap N^d}\xi_{j,x}$. This implies that

$$a(P^{\varphi}_{n,x}) = a(Q^x_{\operatorname{card}(\varphi_n\cap N^d)}).$$

By hypothesis, $\sup_{x\in\Omega} a(Q^x_n)$ vanishes at infinity. However, it is important to note that $\{Q^x_{\operatorname{card}(\varphi_n\cap N^d)}\}$ is not necessarily a subsequence of $\{Q^x_n\}_n$ since $\{\operatorname{card}(\varphi_n\cap N^d)\}_n$ is not necessarily an increasing sequence. Thus, the conclusion does not follow trivially.

We claim that for all $M > 0$ there exists $n_M \in N$ such that $\operatorname{card}(\varphi_n \cap N^d) \geqslant M$ for all $\varphi \in \Psi$ and for all $n \geqslant n_M$. Assume the contrary; then for some $M > 0$ and all $k \in N$ there exist $\varphi^k \in \Psi$ and $n_k \geqslant k$ such that $\operatorname{card}(\varphi^k_{n_k} \cap N^d) < M$. From the assumption that $\sup_{\varphi \in \Psi} m_d(\varphi_n)^{-1} = O(n^{-d})$, it follows that

$$m_d(\varphi_{n_k})^{-1} \operatorname{card}(\varphi^k_{n_k} \cap N^d) = O(n_k^{-d}) \xrightarrow{k} 0.$$

Let $0 < v < 1$. Then, for some $k_0 \in N$ and all $k \geqslant k_0$,

$$v < 1 - m_d(\varphi_{n_k})^{-1} \operatorname{card}(\varphi^k_{n_k} \cap N^d) \leqslant \sup_{\varphi \in \Psi} |m_d(\varphi_{n_k})^{-1} \operatorname{card}(\varphi_{n_k} \cap N^d) - 1|,$$

so we reach a contradiction of the fact that $\sup_{\varphi \in \Psi} |m_d(\varphi_n)^{-1} \times \operatorname{card}(\varphi_n \cap N^d) - 1|$ vanishes at infinity. Thus the claim is true.

Now let $\varepsilon > 0$. Then, there exists $n_0 \in N$ such that $\sup_{x \in \Omega} a(Q^x_n) < \varepsilon$ for all $n \geqslant n_0$. Moreover, according to the previous argument, taking $M = n_0$ there exists some $n_1 \in N$ such that $\operatorname{card}(\varphi_n \cap N^d) \geqslant n_0$ for all $n \geqslant n_1$ and for all $\varphi \in \Psi$. Thus for all $n \geqslant \max\{n_0, n_1\}$ and for all $\varphi \in \Psi$,

$$\sup_{x \in \Omega} a(Q^x_{\operatorname{card}(\varphi_n \cap N^d)}) < \varepsilon.$$

Since this proves that

$$\sup_{\varphi \in \Psi} \sup_{x \in \Omega} a(Q^x_{\operatorname{card}(\varphi_n \cap N^d)}) \to 0,$$

the proof is complete. ∎

Examples and remarks. We now describe some particular cases and examples of these sequence based operators.

(1) If one takes $d = 1$, $\varphi = \{(0, n]\}_n$, $\Psi = \{\varphi\}$, then the Bass–Pyke law becomes the classical strong law of large numbers. In this case sequence based operators are based on arithmetical means of independent and identically distributed random variables.

(2a) For each bounded, continuous $\mathcal{K}(R^p)$ valued function F, one may define the mapping I_F such that $I_F(x) = I_{F(x)}$. Then I_F is bounded and continuous. This provides an example of sequence based operators to set valued functions.

(2b) To each bounded continuous real function f we associate the set valued mapping $t \mapsto \{f(t)\}$. Then its images by sequence based operators are also single valued. Thus sequence based operators can also act on real functions.

(3) If one takes both (1) and (2b) then the sequence based operators are exactly the Feller operators constructed from arithmetical means of i.i.d. random variables [6, Chap. 7; 19; 10].

(4) One may ask if the smooth boundary condition is strong or if, on the contrary, there are large families of sets satisfying it. For instance, any family of convex subsets of $[0, \infty)^d$ does satisfy it [4].

(5) We show now the form of the sequence based versions of Bernstein operators. The functional version of Bernstein operators, which is a particular case of (1) when random variables $\xi_{\bar{1}x}$ are assumed to have a Bernoulli distribution with parameter $x \in [0, 1]$, is defined to be

$$B_n(X, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} X\left(\frac{k}{n}\right)$$

for $x \in [0, 1]$. The general version, given $\varphi \in \Phi$, is

$$B_n^\varphi(X, x) = \sum_{k=0}^{\operatorname{card}(\varphi_n \cap N^d)} \binom{\operatorname{card}(\varphi_n \cap N^d)}{k} x^k (1-x)^{\operatorname{card}(\varphi_n \cap N^d)-k} X\left(\frac{k}{m_d(\varphi_n)}\right).$$

We ought to remark that, in order to ensure that B_n^φ is well defined, one needs to take sequences φ such that $\operatorname{card}(\varphi_n \cap N^d) \leqslant m_d(\varphi_n)$. This limitation does not exist when the interval I is unbounded, as would be the case of Mirakyan-Szász, Baskakov, and many other operators.

(6) It is noteworthy that operators acting on a mapping X converge to co X and not to X itself. This convexification effect is closely related to the behavior of the Aumann integral. In the case of set valued Bernstein operators, it was first shown by Vitale [20].

The following corollary illustrates points (5) and (6), showing that the behavior of sequence based Bernstein operators is essentially the same as in the classical case.

COROLLARY 4.1. *Let $\Omega \subset (0, 1)$ be compact. Let $\Psi \subset \Phi$ satisfy the smooth boundary condition and be such that* $\operatorname{card}(\varphi_n \cap N^d) \leqslant m_d(\varphi_n)$ *for all $\varphi \in \Phi$, and* $\sup_{\varphi \in \Psi} m_d(\varphi_n)^{-1} = O(n^{-d})$. *Then, for each continuous mapping $X: [0, 1] \to \mathscr{F}(R^p)$,*

$$B_n^\varphi(X, \cdot) \xrightarrow{D_\infty} \operatorname{co} X$$

uniformly in $\varphi \in \Psi$, where $D_\infty(X, Y) = \sup_{x \in \Omega} d_\infty(X(x), Y(x))$.

Proof. As we have said, sequence based Bernstein operators are obtained in the case when $\xi_{\vec{1}x}$ has a Bernoulli probability distribution with parameter $x \in [0, 1]$.

It is enough now to check the hypotheses of Theorem 4.2. First, $\sigma^2(x) = x(1-x) \leqslant \frac{1}{4}$ for all $x \in \Omega$, and second,

$$\sup\{Q_n^x(Z) \mid x \in \Omega, Z \text{ is an atom of } Q_n^x\} = \sup_{x \in \Omega} \binom{n}{k} x^k (1-x)^{n-k},$$

which is $o(n^{-1/2})$. ▮

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